$$a\frac{d^2u}{dx^2}+cu=f \qquad u(0)=0,\ a\left.\frac{du}{dx}\right|_{x=L}=Q$$

a, c and f are function of x. here it is assumed that a, c and f are constant on an element.
So, we can write the formulation as follow:

$$\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu=f \qquad u(0)=0,\ a\left.\frac{du}{dx}\right|_{x=L}=Q$$

Weighted residual method is applied to get weak formulation for the given second order differential equation as follow:

A. Weak-Form of the Governing Equations

The solution of the differential equation at a node is approximated by following equation.

$$0=\int_{x_e}^{x_{e+1}} w\left[-\frac{d}{dx}\left(a\frac{d\varphi_h}{dx}\right)+c\varphi_h-q\right]dx$$

$$-w\left[\frac{d}{dx}\left(a\frac{d\varphi_h}{dx}\right)\right]=-\frac{d}{dx}\left(wa\frac{d\varphi_h}{dx}\right)+a\frac{dw}{dx}\frac{d\varphi_h}{dx}$$

$$\Rightarrow\int_{x_e}^{x_{e+1}} w\left[-\frac{d}{dx}\left(a\frac{d\varphi_h}{dx}\right)\right]dx=-\int_{x_e}^{x_{e+1}}\frac{d}{dx}\left(wa\frac{d\varphi_h}{dx}\right)dx+\int_{x_e}^{x_{e+1}}a\frac{dw}{dx}\frac{d\varphi_h}{dx}dx$$

$$=-\left[wa\frac{d\varphi_h}{dx}\right]_{x_e}^{x_{e+1}}+\int_{x_e}^{x_{e+1}}a\frac{dw}{dx}\frac{d\varphi_h}{dx}dx$$

$$\Rightarrow\int_{x_e}^{x_{e+1}} w\left[-\frac{d}{dx}\left(a\frac{d\varphi_h}{dx}\right)+c\varphi_h-q\right]dx=-\left[wa\frac{d\varphi_h}{dx}\right]_{x_e}^{x_{e+1}}+\int_{x_e}^{x_{e+1}}\left[a\frac{dw}{dx}\frac{d\varphi_h}{dx}+cw\varphi_h-qw\right]dx$$

$$=-w\left(x_{e+1}\right)\left.\frac{d\varphi_h}{dx}\right|_{x_{e+1}}+w\left(x_e\right)\left.\frac{d\varphi_h}{dx}\right|_{x_e}+\int_{x_e}^{x_{e+1}}\left[a\frac{dw}{dx}\frac{d\varphi_h}{dx}+cw\varphi_h-qw\right]dx$$

$$\left.\frac{d\varphi_h}{dx}\right|_{x_{e+1}}=Q_{e+1},\left.\frac{d\varphi_h}{dx}\right|_{x_e}=-Q_e$$

$$\int_{x_e}^{x_{e+1}} w\left[-\frac{d}{dx}\left(a\frac{d\varphi_h}{dx}\right)+c\varphi_h-q\right]dx=-w\left(x_{e+1}\right)Q_{e+1}-w\left(x_e\right)Q_e+\int_{x_e}^{x_{e+1}}\left[a\frac{dw}{dx}\frac{d\varphi_h}{dx}+cw\varphi_h-qw\right]dx=0$$

$$B\left(w,u\right)=\int_{x_e}^{x_{e+1}}\left[a\frac{dw}{dx}\frac{d\varphi_h}{dx}+cw\varphi_h-qw\right]dx$$

$$l_e\left(w\right)=-w\left(x_{e+1}\right)Q_{e+1}-w\left(x_e\right)Q_e$$

$$\boxed{\int_{x_e}^{x_{e+1}}\left[a\frac{dw}{dx}\frac{d\varphi_h}{dx}+cw\varphi_h-qw\right]dx=-w\left(x_{e+1}\right)Q_{e+1}-w\left(x_e\right)Q_e}$$

Substitution to transform from physical to natural coordinates

$$\varphi_h(\xi) = \sum_{j=1}^{2} u_j^{\,e} N_j(\xi)$$

$$w = N_j$$

$$\frac{dw}{dx} = \sum_{j=1}^{2} \frac{dN_j}{dx}$$

$$0 = \int_{x_e}^{x_{e+1}} w\left[\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - q\right]dx$$

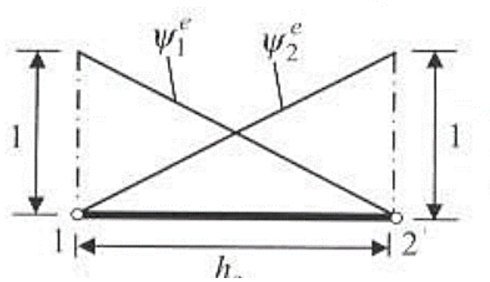
$$0 = \sum_{j=1}^{2}\int_{x_a}^{x_b}\left(a_e u^e \frac{dN_i^{\,e}}{dx}\frac{dN_j^{\,e}}{dx} - c_e \psi_i^{\,e}\psi_j^{\,e} + \psi_i^{\,e} fi\right)dx - \left[\frac{d}{dx}\left(wa\frac{du}{dx}\right)\right]_{x_a}^{x_b}$$

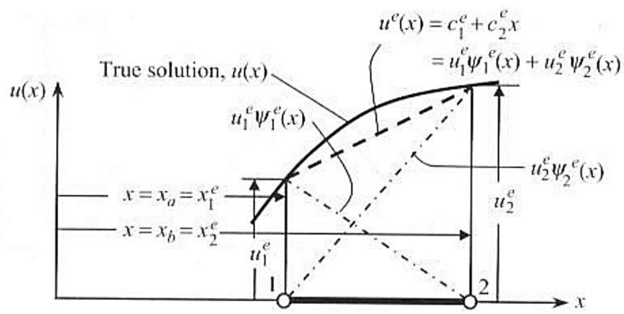
$$\int_{xa}^{xb}\left(a_e u^e \frac{dN_i^{\,e}}{dx}\frac{d\left[N_j^{\,e}\right]}{dx} - c_e N_i^{\,e} N_j^{\,e}\right)dx = K_{ij}$$

$$\int_{xa}^{xb} N_i^{\,e} f_e dx = fi$$

$$\left[K^e\right]\left\{u^e\right\} = \left\{f^e\right\}$$

Linear Element:





$$\psi_1^{\,e} = 1 - \frac{\bar{x}}{h_e}$$

$$\psi_2^{\,e} = \frac{\bar{x}}{h_e}$$

$$\frac{d\psi_1^{\,e}}{d\bar{x}} = -\frac{1}{h_e}$$

$$\frac{d\psi_2^{\,e}}{d\bar{x}} = \frac{1}{h_e}$$

$$K_{ij} = a_e \int_{xa}^{xb}\left(\frac{d\psi_i^{\,e}}{dx}\frac{d\left[\psi_j^{\,e}\right]}{dx}\right)dx$$

So

$$K_{11} = a_e \int_0^{h_e} \left(\frac{d\psi_1{}^e}{dx} \frac{d\left[\psi_1{}^e\right]}{dx} \right) dx = a_e \int_0^{h_e} \left(\frac{-1}{h_e} \frac{-1}{h_e} \right) dx = \frac{a_e}{h_e}$$

$$K_{12} = a_e \int_0^{h_e} \left(\frac{d\psi_1{}^e}{dx} \frac{d\left[\psi_2{}^e\right]}{dx} \right) dx = a_e \int_0^{h_e} \left(\frac{-1}{h_e} \frac{1}{h_e} \right) dx = -\frac{a_e}{h_e}$$

$$K_{21} = a_e \int_0^{h_e} \left(\frac{d\psi_2{}^e}{dx} \frac{d\left[\psi_1{}^e\right]}{dx} \right) dx = a_e \int_0^{h_e} \left(\frac{1}{h_e} \frac{-1}{h_e} \right) dx = -\frac{a_e}{h_e}$$

$$K_{22} = a_e \int_0^{h_e} \left(\frac{d\psi_2{}^e}{dx} \frac{d\left[\psi_2{}^e\right]}{dx} \right) dx = a_e \int_0^{h_e} \left(\frac{1}{h_e} \frac{1}{h_e} \right) dx = \frac{a_e}{h_e}$$

$$\int_0^{h_e} \left(f^e \psi_1{}^e \right) dx = \int_0^{h_e} \left(f^e \left[1 - \frac{\bar{x}}{h_e} \right] \right) dx = \frac{1}{2} f^e h_e$$

$$\int_0^{h_e} \left(f^e \psi_2{}^e \right) dx = \int_0^{h_e} \left(f^e \left[\frac{\bar{x}}{h_e} \right] \right) dx = \frac{1}{2} f^e h_e$$

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}^{(e)} \begin{Bmatrix} u_1{}^e \\ u_1{}^e \end{Bmatrix} = \begin{Bmatrix} f_1{}^e \\ f_1{}^e \end{Bmatrix} + \begin{Bmatrix} Q_1{}^e \\ Q_2{}^e \end{Bmatrix}$$

$$\frac{a_{avg}}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}^{(e)} \begin{Bmatrix} u_1{}^e \\ u_2{}^e \end{Bmatrix} = \begin{Bmatrix} \int_0^{h_e} \left(f^e \psi_1{}^e \right) dx \\ \int_0^{h_e} \left(f^e \psi_2{}^e \right) dx \end{Bmatrix} + \begin{Bmatrix} Q_1{}^e \\ Q_2{}^e \end{Bmatrix}$$

Using the above element matrix,

$$\begin{bmatrix} K_{11}{}^1 & K_{12} \\ K_{21} & K_{22} \end{bmatrix}^{(e)} \begin{Bmatrix} u_1{}^e \\ u_2{}^e \end{Bmatrix} = \begin{Bmatrix} f_1{}^e \\ f_2{}^e \end{Bmatrix} + \begin{Bmatrix} Q_1{}^e \\ Q_2{}^e \end{Bmatrix}$$

$$\begin{bmatrix} K_{11}{}^1 & K_{12}{}^1 & 0 & 0 & 0 \\ K_{21}{}^1 & K_{22}{}^1 + K_{11}{}^2 & K_{12}{}^2 & 0 & 0 \\ 0 & K_{21}{}^2 & K_{22}{}^2 + K_{11}{}^3 & K_{12}{}^3 & 0 \\ 0 & 0 & K_{21}{}^3 & K_{22}{}^3 + K_{11}{}^4 & K_{12}{}^4 \\ 0 & 0 & 0 & K_{21}{}^4 & K_{22}{}^4 \end{bmatrix} \begin{Bmatrix} U^1 \\ U \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_1^2 + f_2^1 \\ f_2^2 + f_1^3 \\ f_2^3 + f_1^4 \\ f_2^4 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_1^2 + Q_2^1 \\ Q_2^2 + f_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ K_{21} & K_{22}+K_{11} & K_{12} & 0 & 0 \\ 0 & K_{21} & K_{22}+K_{11} & K_{12} & 0 \\ 0 & 0 & K_{21} & K_{22}+K_{11} & K_{12} \\ 0 & 0 & 0 & K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} U^1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_1^{\,2} + f_2^{\,1} \\ f_2^{\,2} + f_1^{\,3} \\ f_2^{\,3} + f_1^{\,4} \\ f_2^{\,4} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ Q \end{Bmatrix}$$

General Equation, obtained

$$\frac{a_{avg}}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}^{(e)} \begin{Bmatrix} u_1^{\,e} \\ u_2^{\,e} \end{Bmatrix} = \begin{Bmatrix} \int_0^{h_e} \left(f^{\,e} \psi_1^{\,e} \right) dx \\ \int_0^{h_e} \left(f^{\,e} \psi_2^{\,e} \right) dx \end{Bmatrix} + \begin{Bmatrix} Q_1^{\,e} \\ Q_2^{\,e} \end{Bmatrix}$$